On equivariant Euler characteristics

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Abstract. *The Euler characteristic of an orbifold M/G as used in string theory is identified with the Euler characteristic of equivariant K-theory $K_G(M)$.*

INTRODUCTION

This is an expository note drawing the attention of physicists to the relevance of *equivariant K-theory* for certain topics in quantum field theory. Our main observation, concerning Euler characteristics, has been made independently by A. Connes, M. Hopkins (and probably others), but since equivariant K-theory is unfamiliar to most physicists it is perhaps useful to publicize the situation.

In the physics of string theory the Euler characteristic of the target manifold (where the string moves) plays an important role, related to the number of «generations». When the (compact) manifold M is replaced by a quotient M/G, where G is a finite group of symmetries, it was noted [7] that the correct Euler characteristic for string theory was the expression:

$$\chi(M,G) = \frac{1}{|G|}\sum_{g_1,g_2}{}' \chi(M^{g_1,g_2}) \tag{1.1}$$

where the sum Σ' is over *commuting pairs* g_1, g_2 in G, M^{g_1,g_2} is the simultaneous fixed-point set of g_1, g_2 and $|G|$ is the order of G. For a free action there is only one

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term in the sum and

$$\chi(M,G) = \frac{1}{|G|}\chi(M) = \chi(M/G) \tag{1.2}$$

coincides with the Euler characteristic of the manifold M/G. For general group actions the three terms in (1.2) are all different, and from the point of view of standard algebraic topology it is not clear what special significance there is in $\chi(M,G)$ as defined by (1.1).

The purpose of this note is to give a simple interpretation of $\chi(M,G)$ in terms of *equivariant* K*-theory*. The result in question (Theorem 1 below) is in essence an elementary consequence of known facts.

We recall [7] [11] that $K_G^0(M)$ is defined as the abelian group generated by complex G-vector bundles over M (i.e. vector bundles endowed with a G-action covering the action on M). Because the Bott periodicity theorem extends to the equivariant case [11] we can extend $K_G^0(M)$ to a Z_2-graded cohomology theory

$$K_G^*(M) = K_G^0(M) \otimes K_G^1(M).$$

The group $K_G^1(M)$ can be explicitly defined as the kernel of the restriction map

$$K_G^0(M \times S^1) \to K_G^0(M)$$

given by the inclusion of a point in S^1.

The groups $K_G^0(M), K_G^1(M)$ are finitely-generated and so each has a rank ($=$ dim $K_G \otimes Q$). Our result is then:

THEOREM 1. $\chi(M,G) =$ *rank* $K_G^0(M) -$ *rank* $K_G^1(M)$.

REMARK. When G acts freely,

$$K_G^*(M) \cong K^*(M/G).$$

Moreover the Chern character induces a Z_2-graded isomorphism

$$K^* \otimes Q \to H^*(\ ,Q)$$

so that the theorem in this case just reduces to

$$\chi(M,G) = \chi(M/G).$$

Theorem 1, which deals with ranks (or dimensions) is a consequence of a theorem relating $K_G^*(M)$ to the K-groups of various fixed-point sets. The prototype of such results is the localization theorem of [3] which is extensively used in various «Lefschetz formulas» for elliptic operators. The general result we need is

THEOREM 2. *There is a natural isomorphism*

$$K_G^*(M) \otimes C \cong \bigoplus_{[g]} [K^*(M^g) \otimes C]^{Z_g}$$

where Z_g *is the centralizer of* g *in* G, *the sum is over conjugacy classes* $[g]$ *in* G *and* $[\ ]^{Z_g}$ *denotes the* Z_g*-invariant part.*

REMARK. Recall that $K_G^*(M)$ is naturally a module over the representation or character ring

$$R(G) = K_G^0 \text{ (point)}.$$

After tensoring with C we can identity this with the ring of class functions on G. Thus $K_G^*(X) \otimes C$ naturally breaks up as a direct sum over conjugacy classes. Theorem 2 asserts that the component (or localization) at $[g]$ is the Z_g-invariant part of $K^*(M^g) \otimes C$.

2. PROOFS

To prove Theorem 2 we first define homomorphisms

$$\phi_g : K_G^*(M) \otimes C \to [K^*(M^g) \otimes C]^{Z_g}$$

as follows. If E is a G-vector bundle over M its restriction to M^g is acted on fibrewise by g and so decomposes as a direct sum of subbundles E_ζ for each eigenvalue ζ of g. Put

$$\phi_g(E) = \sum_\zeta \zeta E_\zeta.$$

For K^1 replace M by $M \times S^1$.

We next observe that both terms in Theorem 2 are Z_2-graded periodic cohomology theories on the category of compact G-spaces, and that

$$\phi = \bigoplus_{[g]} \phi_g$$

is a natural homomorphism. Using the general machinery developed in [11] it is then sufficient to check Theorem 2 when $M = G/H$ is a (finite) homogeneous space. In this case

$$K_G^*(G/H) \otimes C \cong R(H) \otimes C$$

$$(G/H)^g/Z_h = H\text{-conjugacy classes contained in } [g]$$

and ϕ is just the character isomorphism.

We shall now make a number of additional remarks about Theorem 2 and its proof which may be helpful.

(1) The localization theorem of [3] asserts that, for a conjugacy class γ in G, the γ-component of $K_G^*(M) \otimes C$ is isomorphic to the γ-component of $K_G^*(M^\gamma) \otimes C$, where

$$M^\gamma = \bigcup_{g \in \gamma} M^g .$$

In view of this, Theorem 2 is equivalent to the assertion that the map

$$\coprod_{g \in \gamma} M^g \to \bigcup_{g \in \gamma} M^g , \tag{2.1}$$

from the *disjoint* union $\coprod$ to the ambient union in M, induces an isomorphism for the γ-component of K_G^*. This is entirely obvious if the M^g are disjoint in M, and it is somewhat surprising that it continues to hold in general.

(2) Using the Chern character isomorphism for M^g we can replace $K^*(M^g) \otimes C$ by $H^*(M^g, C)$. To compute the Lefschetz number $L(h)$ of an element $h \in Z_g$ on this we can apply the Lefschetz fixed-point formula (allowing for higher-dimensional fixed point sets) and get

$$L(h) = \chi(M^{g,h}) .$$

Standard character theory then leads to the dimension formula:

$$[\dim K^0(M^g) \otimes C]^{Z_g} - \dim[K^1(M^g \otimes C)]^{Z_g} =$$

$$= \frac{1}{|Z_g|} \sum_{h \in Z_g} \chi(M^{g,h}) .$$

Together with Theorem 2 this leads at once to Theorem 1, provided we now sum over elements g, rather than conjugacy classes $[g]$, and note that $|G|/|Z_g|$ is the number of such classes.

(3) The right side of the isomorphism in Theorem 2, with $K^* \otimes C$ replaced by $H^*(\ ,C)$ is the definition of *de-localized equivariant cohomology* in the sense of [4] [5]. Theorem 2 is then viewed as a generalized Chern character isomorphism and is proved in [5] by the methods indicated above. Since the proof is given in full detail in [5] we have been brief in our presentation. The methods are in any case quite standard. In [5] the main emphasis is on infinite discrete groups.

(4) In string theory the right side of (1.1) arises naturally as a sum over conjugacy classes $[g]$. Each term arises from the space of paths

$$f : \{0,1\} \to M, \qquad f(1) = gf(0).$$

Theorem 2 really identifies the contribution of each conjugacy class separately.

3. FURTHER COMMENTS

One of the clues that suggested Theorem 1 is the fact that the relevant «Hodge theory» for a loop space is only Z_2-graded and not Z-graded. This was explained by Witten in [12] and was further analysed in relation to S^1-equivariant cohomology in [2]. The Z_2-grading suggests K-theory rather than cohomology as the natural framework, and this acquires real significance in the G-equivariant case (for finite G) as shown by Theorem 1.

Another clue was provided by the finite subgroups G of $SU(2)$ and the «Kleinian singularities» to which they give rise.

In fact if M is an algebraic surface with an action of G having isolated fixed points P (with the standard action locally on C^2) then M/G has singularities which have a well-known standard resolution by a graph of ℓ exceptional (rational) curves [6]. Let $(M/G)'$ denote the new surface with all singularities resolved. Then Theorem 2 leads to the following formula

$$\chi(M,G) = \chi((M/G)'). \tag{3.1}$$

To see this we note first that each fixed point P in M contributes

$$\begin{aligned} K_G^*(P) &= R(G) \quad \text{in} \quad K_G\text{-theory} \\ H^*(P,Z) &= Z \quad \text{in integral cohomology.} \end{aligned}$$

This implies that

$$\chi(M,G) = \chi(M/G) + N(c(G) - 1)$$

where N is the number of fixed points and $c(G)$ is the number of conjugacy classes of G. On the other hand, from the resolution

$$\chi((M/G)') = \chi(M/G) + N\ell.$$

But, by a result of McKay [10]

$$c(G) = \ell + 1$$

which establishes (3.1). As a famous example take M to an abelian surface (complex torus), $G = Z_2$. Then $N = 16$, $\ell = 1$ and $(M/G)'$ is diffeomorphic to a $K3$ surface which has $\chi = 24$. It was well known to physicists that (3.1) held in many special cases including the $K3$ surface.

In homotopy theory Hopkins and Kuhn [8] have investigated a hierarchy of generalized cohomology theories (beginning with $H^*, K^*, \ldots$) and have found formulas involving n-tuples of commuting elements of G at the nth level of the hierarchy. It is not clear what precise relation their work has to the physics of loop spaces or the Euler characteristics studied here. Note however that the two formulae

$$\chi(M/G) = \frac{1}{|G|}\Sigma_g\, \chi(M^g) \qquad \text{(Lefschetz formula)}$$

$$\chi(M,G) = \frac{1}{|G|}\sum_{g_1,g_2}{}' \chi(M^{g_1,g_2}) \qquad \text{(Theorem 1)}$$

suggest that one might consider

$$\frac{1}{|G|}\sum_{g_1,g_2,g_3}{}'' \chi(M^{g_1,g_2,g_3})$$

summed over *commuting triples*, and that this might have something to do with G-equivariant *elliptic cohomology* (which is essentially the next theory in the hierarchy).

Finally we would like to draw attention to a very interesting commutative algebra, derived from equivariant K-theory, which has been introduced by Lusztig [9]. This is $K_G(G) \otimes C$ where G acts on itself by conjugation. The multiplication in this algebra is induced by the multiplication in G. It has been pointed out to us by R. Dijkgraaf and others that Lusztig's algebra can be interpreted as the Verlinde algebra (at level zero) for the finite group G [13], [14]. It seems therefore that K_G-theory appears very naturally in conformal field theory, at least when G is finite. It would be interesting to investigate its role in greater detail and in particular to consider K_G for compact (non-finite) Lie groups.

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